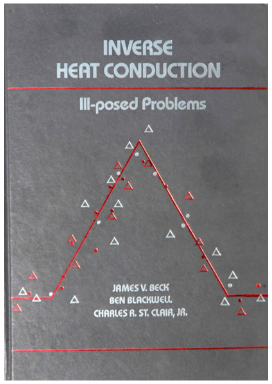
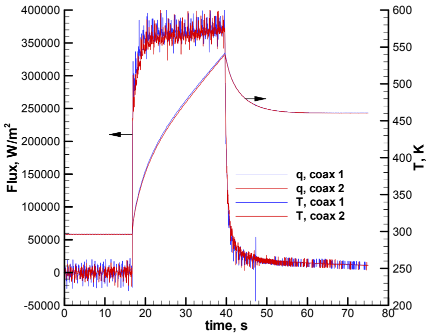
Blackwell Consulting
PO Box 2879
Corrales, NM 87048
bblackwell13@comcast.net
505-897-5090
fax: 505-890-4992





A Technique for Uncertainty Analysis for Inverse Heat Conduction Problems

Ben Blackwell, Blackwell Consulting
Jim Beck, Michigan State University

Presented at 2009 Inverse Problems Symposium

(www.inverseproblems2009.org)

May 31-June 2, 2009

Michigan State University

East Lansing, MI 48824

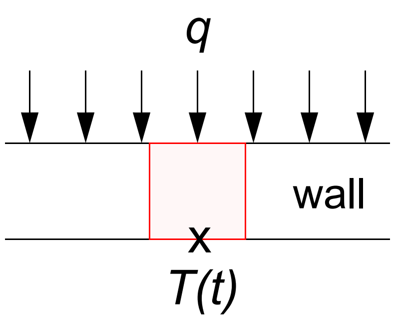
Introduction

- Inverse Heat Conduction Problem (IHCP) is determination of heat flux from interior temperature measurements (not necessary for this group)
- In today's environment, it is not adequate to just run your favorite IHCP algorithm and report the results
- Instead, we need to be concerned with the uncertainty in the reported results
- Ingredients of IHCP solution are:
    - Computational model with uncertain input parameters
    - Experimental temperature data with measurement uncertainty
- Question we will address is the determination of the uncertainty in the estimated heat flux, given the uncertainty in input parameters and experimental temperature measurements

Outline of Talk

- Math model
- IHCP algorithm description (brief)
- Propagation of variance equation
- Filter coefficient form of IHCP algorithm
- Parameter importance factors
- Example calculations
- Thoughts on extending analysis to nonlinear problem of temperature dependent properties
- Summary
- Acknowledgements

Math Model



$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)$$

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q(t)$$

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = q_b(t) \text{ or } T(L,t) = T_b$$

$$T(x,0) = T_i(x)$$

- **We assumed constant thermal properties *k* and ρ*c* and they have uncertainty associated with them**
- **Temperature measurements $T(x_j,t_i) = Y_{j,i}$ have uncertainty**

Blackwell Consulting

IHCP Algorithm Description

- **Algorithm will minimize sum of squares function**

$$S = \sum_{i=1}^{r} \sum_{j=1}^{J} (Y_{j,M+i-1} - T_{j,M+i-1})^2$$

- **We selected the Duhamel's theorem based function specification method given in Beck, Blackwell and St. Clair Jr.**

$$q_M = \frac{\sum_{i=1}^{r} \sum_{j=1}^{J} (Y_{j,M+i-1} - \hat{T}_{j,M+i-1}) \phi_{j,i}}{\sum_{i=1}^{r} \sum_{j=1}^{J} \phi^2{}_{j,i}}, M = 1, 2, \ldots, M_{max}$$

$$\hat{T}_{j,M+i-1}\Big|_{q_m = q_{M+1} = \ldots = q_{M+r-1} = 0} = T_o + \sum_{n=1}^{M-1} q_n \Delta\phi_{M-n+r-1}$$

$$\Delta\phi_i = \phi_{i+1} - \phi_i$$

Blackwell Consulting

IHCP Algorithm (con't)

- **Sensitivity coefficients $\phi = \partial T / \partial q$ are computed from (p. 14 Beck, Blackwell, St. Clair Jr.)**

$$\frac{T - T_o}{q_c L / k} = \frac{\alpha t}{L^2} + \frac{1}{3} - \frac{x}{L} + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-n^2 \pi^2 \frac{\alpha t}{L^2}\right) \cos\left(n\pi \frac{x}{L}\right), \alpha = k/C$$

- **In this instance,**

$$\phi(x, t) = \frac{L}{k}\left[\frac{\alpha t}{L^2} + \frac{1}{3} - \frac{x}{L} + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-n^2 \pi^2 \frac{\alpha t}{L^2}\right) \cos\left(n\pi \frac{x}{L}\right)\right]$$

- **$\phi(x, t)$ can be computed for other geometries (e.g. cylindrical or spherical)**
- **Fully numerical procedures such as finite difference, finite element or finite volume could be used**
- **Above analytical equation for planar geometry was used for this work**

Propagation of Variance Equations

- **Functional dependence of data reduction equation (properties and temperature measurements)**

$$q_M = q_M\left(k, \rho c, Y_o, \underbrace{Y_{1,1}, Y_{1,2}, \ldots, Y_{1,M+r-1}}, \underbrace{Y_{2,1}, Y_{2,1}, \ldots, Y_{2,M+r-1}} \ldots\right)$$

- **Uncertainty propagation equation**

$$\sigma^2_{q_M} = \left(\frac{\partial q_M}{\partial k}\sigma_k\right)^2 + \left(\frac{\partial q_M}{\partial \rho c}\sigma_{\rho c}\right)^2 + \left(\frac{\partial q_M}{\partial Y_o}\sigma_{Y_o}\right)^2 + \sum_{j=1}^{J}\sum_{i=1}^{M+r-1}\left(\frac{\partial q_M}{\partial Y_{j,i}}\sigma_{Y_{j,i}}\right)^2$$

- **Data reduction equation (IHCP algorithm) is used to compute sensitivity coefficients in uncertainty propagation equation**
  - **This differentiation for sensitivity of the estimated heat flux with respect to the individual measurements is easier if the filter coefficient form of the IHCP algorithm is used**

Filter Coefficient form of IHCP

- **From page 148 of Beck, Blackwell and St. Clair Jr,**

$$q_M = \sum_{j=1}^{J} \sum_{n=1}^{M+r-1} g_{j,n}(Y_{j,M-n+r} - Y_o) = \sum_{j=1}^{J} \sum_{i=1}^{M+r-1} g_{j,M+r-i}(Y_{j,i} - Y_o)$$

- **Measurement sensitivity coefficients are**

$$\frac{\partial q_M}{\partial Y_{j,i}} = g_{j,M+r-i}$$

$$\frac{\partial q_M}{\partial Y_o} = -\sum_{j=1}^{J} \sum_{n=1}^{M+r-1} g_{j,n} = -\sum_{j=1}^{J} \sum_{i=1}^{M+r-1} g_{j,M+r-i}$$

- **For adiabatic back boundary condition**

$$\lim_{M \to \infty} \sum_{j=1}^{J} \sum_{n=1}^{M+r-1} g_{j,n} = 0$$

- **For sufficiently large M, error in Y_o becomes unimportant**

Sensitivity Coefficients for Thermal Properties

- Second order central differences were used for $\partial q_M / \partial k$ and $\partial q_M / \partial C$

$$\frac{\partial q_M}{\partial k} = \frac{q_M(k + \Delta k, \rho c, Y_o, Y_{1,1}, Y_{1,2}, \ldots) - q_M(k - \Delta k, \rho c, Y_o, Y_{1,1}, Y_{1,2}, \ldots)}{2\Delta k}$$

- We prefer to work with scaled sensitivity coefficients

$$k\frac{\partial q_M}{\partial k} = \frac{q_M(k(1 + \delta), \rho c, Y_o, Y_{1,1}, Y_{1,2}, \ldots) - q_M(k(1 - \delta), \rho c, Y_o, Y_{1,1}, Y_{1,2}, \ldots)}{2\delta}$$

$$\delta = \frac{\Delta k}{k} = \frac{\Delta C}{C} \text{ with } 10^{-6} < \delta < 10^{-3}$$

- Similar finite difference equations were used for $C\partial q_M / \partial C$

Parameter Importance Factors

- **How do we assign a relative importance to various parameters when uncertainty is present?**

- **Divide propagation of variance equation through by $\sigma^2_{q_M}$**

$$1 = \frac{1}{\sigma^2_{q_M}}\left(k\frac{\partial q_M}{\partial k}\frac{\sigma_k}{k}\right)^2 + \frac{1}{\sigma^2_{q_M}}\left(\rho c\frac{\partial q_M}{\partial \rho c}\frac{\sigma_{\rho c}}{\rho c}\right)^2 + \frac{1}{\sigma^2_{q_M}}\left(\Delta T\frac{\sigma_{Y_o}}{\Delta T}\sum_{j=1}^{J}\sum_{n=1}^{M+r-1} g_{j,n}\right)^2 +$$

$$\frac{1}{\sigma^2_{q_M}}\left(\sum_{j=1}^{J}\sum_{i=1}^{M+r-1}\left(\Delta T g_{j,M+r-i}\frac{\sigma_{Y_{j,i}}}{\Delta T}\right)\right)^2$$

- **Terms on right hand side are importance factors for *k*, ρc, Y_o and all other measured temperatures respectively**

Blackwell Consulting

Algorithm for Computing Filter Coefficients

- **Non filter coefficient form of IHCP algorithm is used to compute filter coefficients**
- **For N measurement times, we need N filter coefficients for each sensor location**
- **Since filter coefficients decrease to zero for large N, fewer than N filter coefficients may be adequate**
- **Steps are as follows:**

1. **For sensor j, set measurement vector $Y_{j,i} = 0$, $i=1,...,N$ except for $Y_{j,r} = 1$**
2. **Apply non filter form of IHCP algorithm to compute sequence of heat fluxes q_M, $M = 1,2,...,N$**
3. **For sensor j, the filter coefficients are $g_{j,i} = q_i$, $i = 1,2,...,N$**
4. **Repeat steps 1-3 for each additional value of j**

Example Calculation

- Representative of laser interaction with stainless steel plate
- Analytical solution used to generate “data” with random noise added
- Problem parameters are:

$$L = 0.0015 \text{ m}$$

$$k = 18 \text{ W/m-k}$$

$$\alpha = 4.39\text{x}10^{-6} \text{ m}^2/\text{s}$$

$$q_c = 3.54\text{x}10^{6} \text{ W/m}^2$$

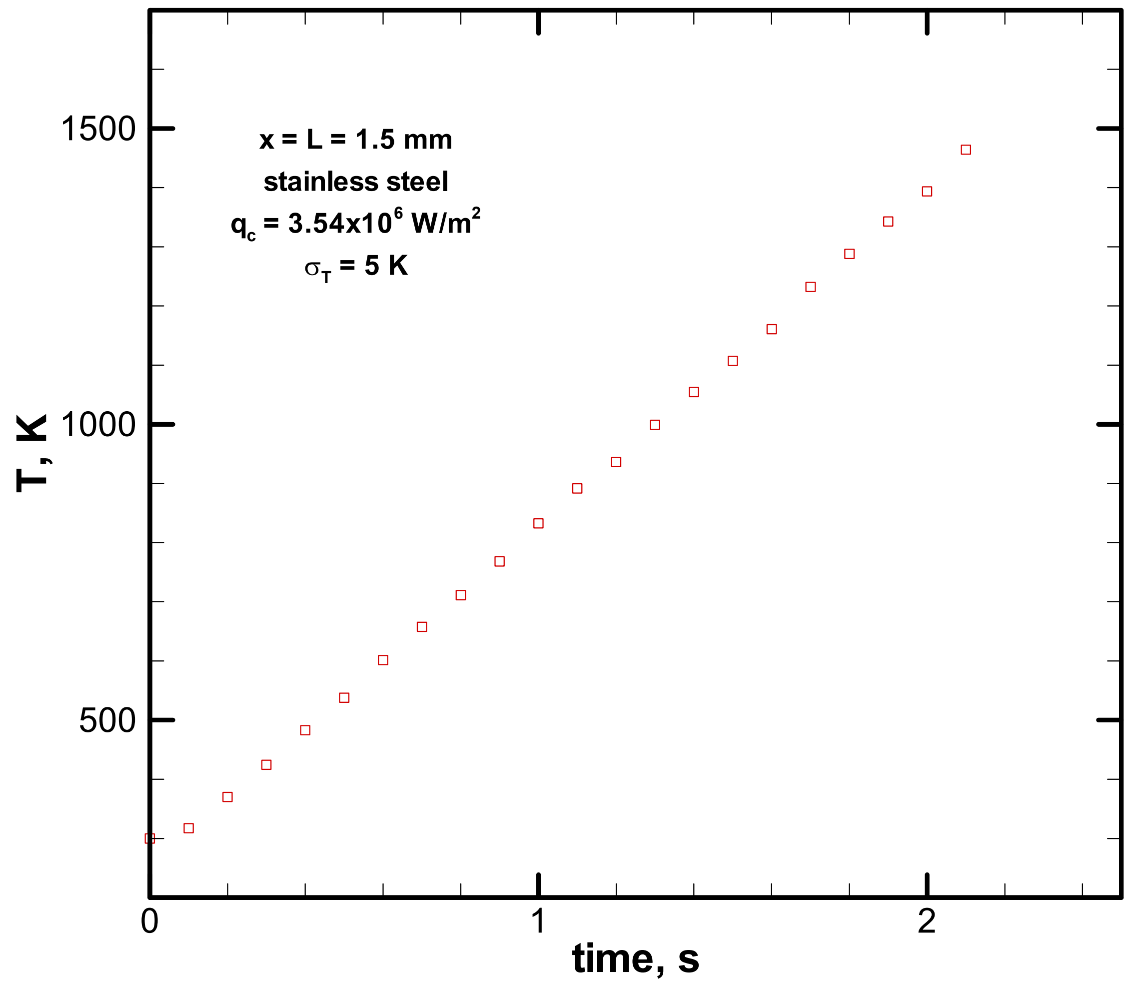
$$\Delta t = 0.1 \text{ s}$$

$$\sigma_k/k = 0.1$$

$$\sigma_C/C = 0.05 \ (C = \rho c)$$

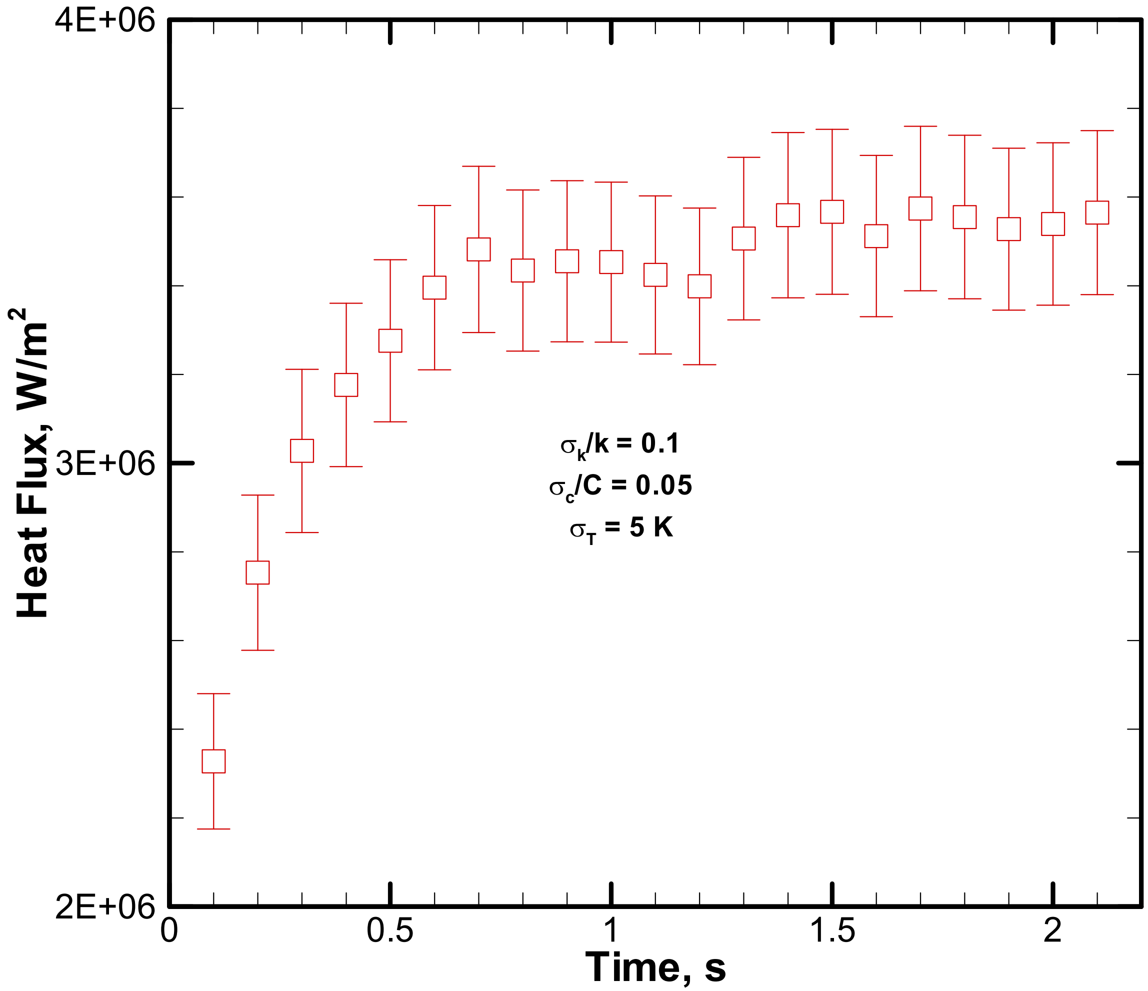
$$\sigma_Y = 5 \text{ K}$$

Back Face Temperature Data (with noise)



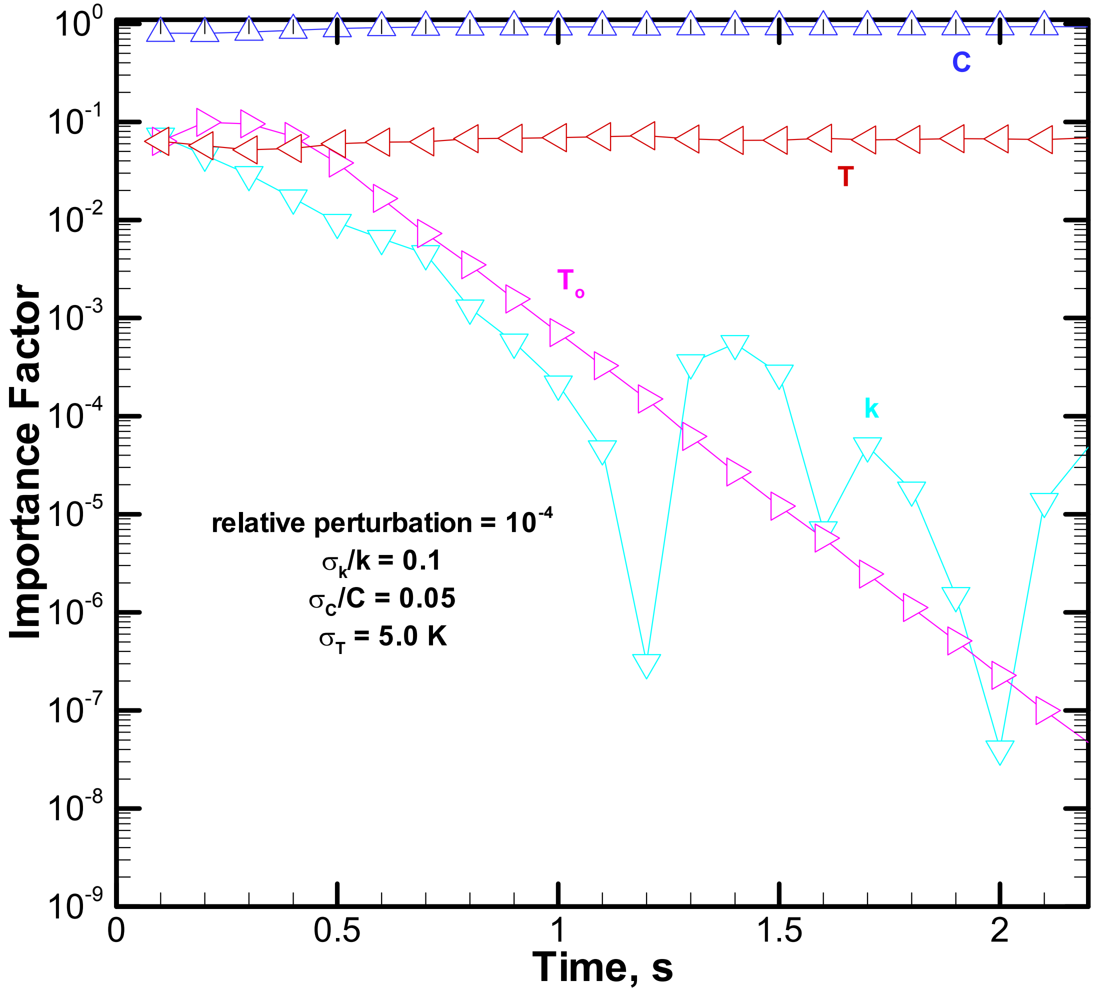
Blackwell Consulting

Estimated Heat Flux with Uncertainty Bars(r = 5)

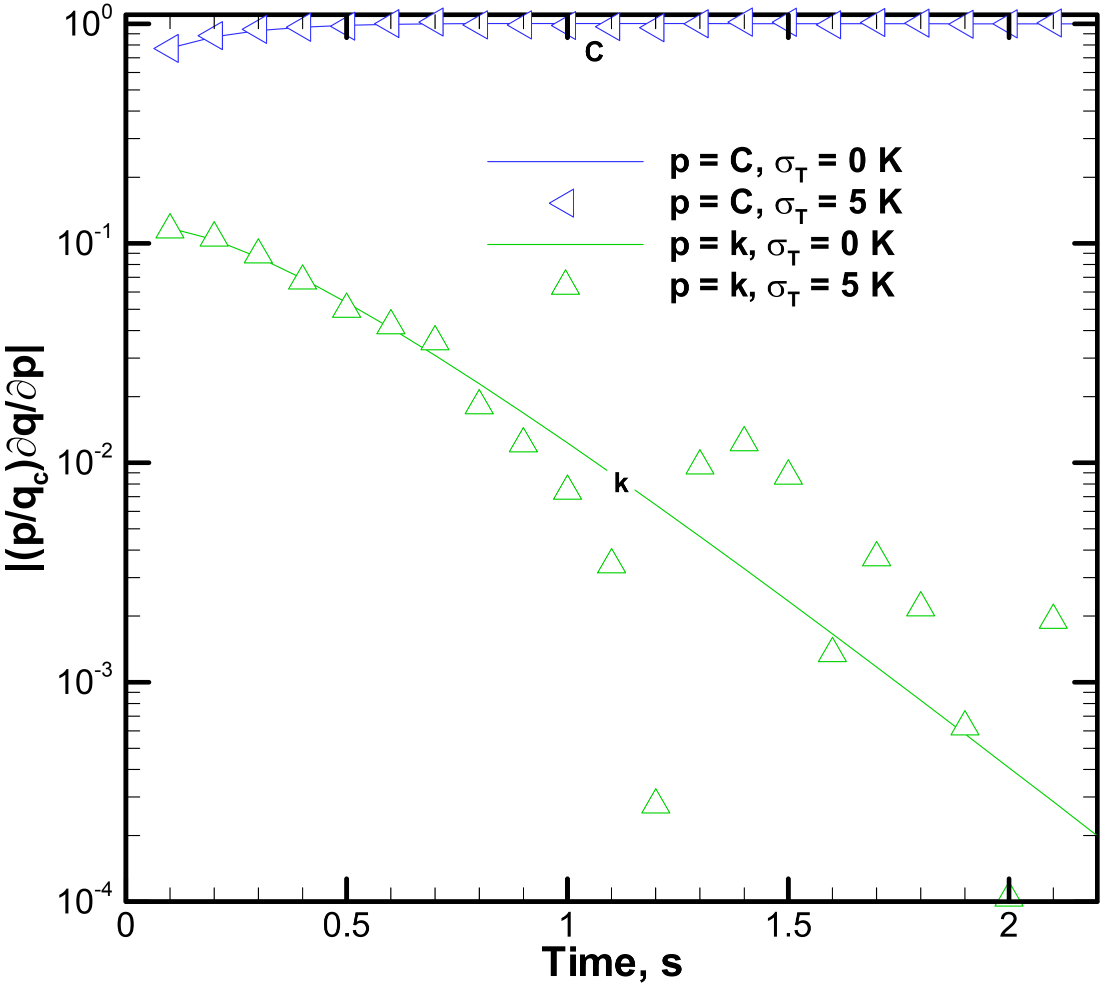


Blackwell
Consulting

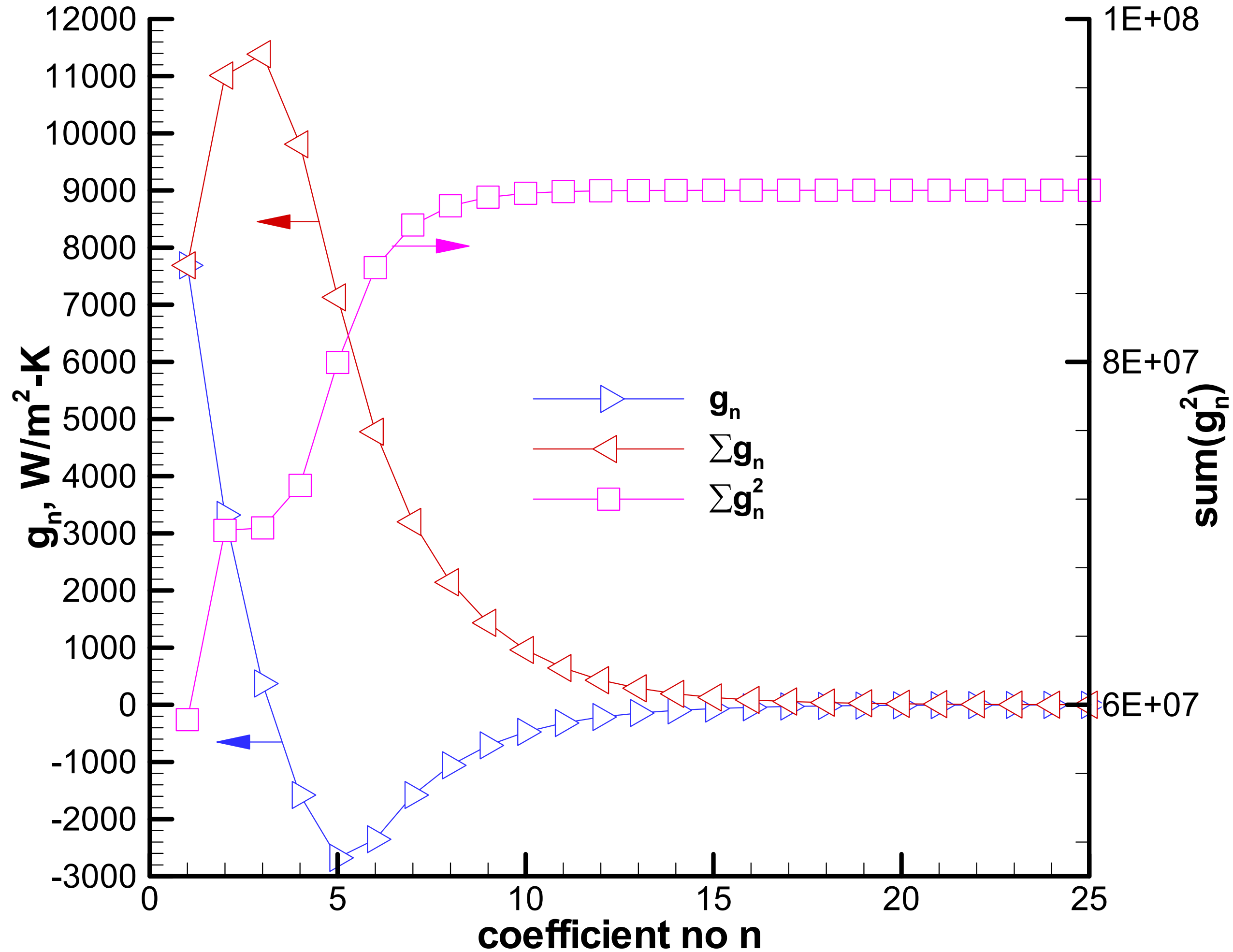
Importance Factors



Property Sensitivity Coefficients



Functions of Filter Coefficients



Blackwell Consulting

Thoughts on Extension to Non Linear Problems

- Filter form of IHCP algorithm is restricted to constant properties
- Professor Beck has introduced an approximate technique of evaluating the properties at an average plate temperature and computing corresponding filter coefficients
- This produces a set of filter coefficients that are temperature dependent
- For the example presented here, temperature drop across plate is small in comparison to temperature level
- This would allow one to interpolate in temperature for appropriate set of temperature dependent filter coefficients

Summary

- **Filter coefficient form of IHCP was implemented along with uncertainty in estimated heat flux due to uncertainties in thermal properties and temperature measurements**
- **Importance factors were introduced and the volumetric heat capacity was the dominant contributor to the overall uncertainty in estimated heat flux for example presented**
  - **Some analytical results (not presented) explain why volumetric heat capacity is dominant parameter**
- **Uncertainty analysis provides a road map on how to best reduce the overall uncertainty in estimated heat flux**
  - **Focus on those parameters with largest importance factors**

Acknowledgements

The authors would like to thank the Test Resource Management Center (TRMC) Test and Evaluation/Science & Technology (T&E/S&T) Program for their support. This work is funded by the T&E/S&T Program through the U.S. Army Program Executive Office for Simulation, Training and Instrumentation's (PEO STRI) contract number W900KK-08-C-0008.